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A METHOD OF SHAKEDOWN ANALYSIS OF FRAMES AND ARCHES

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Abstract—Structures, in general, are subjected to more complex load programs than those considered in the theory of limit analysis. The shakedown theory presents general theorems allowing to estimate does the structure adapt to the given load program or not. On the basis of previous general results [22] the present paper gives an approximate method of analysis of frames and arches of arbitrary solid cross-section in the case when the influence of axial forces on the stress state cannot be neglected. The problem reduces to the linear programming problem. Numerical results for a portal frame and for a circular arch of rectangular cross-section are presented.

1. INTRODUCTION

THE adaptation of elastic–plastic structures to prescribed loads varying in time has been considered first in the late twenties [1, 2]. Melan [3], Koiter [5], Prager [6], Rozenblum [7, 8] and others presented some more or less rigorously formulated criteria of shake-down for the general case of the elastic–plastic continuum.

Structures in general are subjected to more complex load programs than those considered in the theory of limit analysis where all loads are assumed to depend proportionally on a single parameter varying monotonically with time.

If an elastic–plastic structure is subjected to loads varying with time, there is a danger of collapse before the limit load interaction surface is reached. This collapse is due to gradually increasing plastic deformations (incremental collapse) or to alternating plastic deformations resulting in a plastic fatigue (low cycle fatigue). It is however also possible that plastic deformations arising in the initial cycles of loading give such a field of residual stress that the response of the structure to subsequent cycles will be purely elastic.

Melan [3] and Koiter [5] formulated two theorems enabling to estimate whether the structure does shake down to the given load program or not. However, the direct use of those theorems in more complex cases is hindered by computational difficulties. Moreover the engineering theories of structures (theory of plates and shells, theory of beams, etc.) are expressed in terms of appropriate generalized variables rather than in terms of stresses and strains. It appeared worth while, therefore, to investigate how these theorems might be expressed in generalized quantities (see [22]). This can be done easily in the case of sandwich structures, as it has been shown for a circular arch in Ref. [14]. Some authors merely assume the response of a cross-section to be purely elastic when the generalized stresses remain within the yield locus (see [13, 18]). This assumption is quite reasonable for arches and frames of I section but may be risky in case of other cross-sections. On the other hand if we assume that a cross-section may respond elastically only within the range of its initial elastic domain—as it was done in Refs. [20] and [21]—then we neglect the fact that residual stresses may change the range of perfectly elastic response of the cross-section. It was shown in Ref. [10] that at pure bending of a beam the elastic domain shifts within certain limits.

In Ref. [22] a method was proposed enabling to apply the Melan theorem when the theory regarding the structure under consideration employs generalized stresses. Such a procedure is applied to elastic–plastic plates in Ref. [23].

It was shown in [22] that Melan's theorem regarding the adaptation of a structure to a given load program can be expressed in the following equivalent form:

a structure will shake down to the given load program if there exists a time-independent field Q_r^0 of generalized residual stresses (i.e. satisfying the equilibrium equations for vanishing loads) and if there exists, for every section ξ of the structure, an elastic locus S_ξ such that the sum

$$Q_r = Q_r^0 + Q_r^e$$

will remain within the elastic locus S.

Here Q_r stands for the actual generalized stress field. Q_r^e denotes the stress for a geometrically identical and identically loaded structure of unlimited elastic response. An elastic locus is defined as a domain in the space of the generalized stresses, within which the response at any point of the section remains perfectly elastic. Certain general properties of elastic loci were presented in Ref. [22] and some of them will be used in the present paper.

The main idea is the next. Melan's theorem can be stated as follows:

a structure shakes down to the prescribed loading program if there exists such an independent of time state of internal stresses which is in equilibrium with one of the possible states of load and if any variation of the load around this particular state does not produce plastic deformations.

The generalized theorem gives an estimate of adaptability of the structure within the frames of accuracy of an applied theory describing the stress state in terms of generalized stresses.

Obviously such a method does not pretend to give solutions which are exact in the sense of a three dimensional theory yet it seems to be an improvement with respect to the hitherto results.

It appears that the concept of elastic loci proves useful in applications. Hodge and Kalinowski [14] presented an analysis of shakedown of a circular arch with an ideal sandwich cross-section. Using the idea of elastic loci, the present paper develops a method of approximate solutions for arches and frames subjected to heavy axial loads. The forms of the cross-sections may be arbitrary. This seems to be of importance because, as will be shown, the boundary of a shakedown domain for sandwich arch does not necessarily need to constitute either an upper or a lower bound of the shakedown domain of arches of uniform cross-sections. Attention is focused on the construction of the simplest possible elastic loci using the simplest distribution of residual stresses.

For definiteness it will be assumed that the external loads the structure is subjected to depend linearly on a set of r parameters: $p_1, p_2, \ldots, p_r$. Hence the solution of an elastic response must have the following form:

$$M^e(\xi) = \sum_{i=1}^{r} p_i M^i(\xi); \qquad N^e(\xi) = \sum_{i=1}^{r} p_i N^i(\xi) \tag{1.1}$$

where ξ numerates the cross-sections of the structure.

The load program is assumed in the form:

$$\delta_i p_i^s < p_i < p_i^s; \qquad i = 1, 2, \ldots, r \tag{1.2}$$

each load varies independently within the above specified limits.

Now, let the coefficients δ_i be given and let us regard the r-dimensional Euclidean space of the load parameters p_i. Let the set of all points $(p_1^s, p_2^s, \ldots, p_r^s)$ such that the structure shakes down to the load program (1.2) be called the shakedown domain and its boundary the shakedown interaction surface (curve).

If the yield condition of the material of the structure is convex, then any shakedown domain must also be convex (see [14] and further [22] where this requirement may be obtained from the theorems presented therein).

2. PROPERTIES OF ELASTIC LOCI

In the bending theory of frameworks and arches there are two generalized stresses acting at any cross-section, namely the bending moment M and the axial force N. An elastic locus then constitutes an area in the (M, N) plane. Obviously, any elastic locus for the cross-section is contained within the yield locus of that section.

Let us consider a cross-section as in Fig. 1, with one vertical axis of symmetry, of depth $h = H^- + H^+$, of width $B(z)$ and of area A; its inertia moment is denoted by J. The origin of the coordinate system coincides with the center of gravity of the section. The form of an elastic locus depends on the form of the cross-section as well as on the field of the residual stress $\rho(z)$ resulting from the previous plastic deformation of the cross-section.

To construct an elastic locus associated with the residual stress distribution $\rho(z)$ we assume that the cross-section responds elastically to the given stress resultants and therefore the elastic part of the stress is

$$\sigma^e(z) = \frac{M}{J}z + \frac{N}{A} \tag{2.1}$$

according to the elementary formula of strength of materials. The condition

$$|\sigma^e(z) + \rho(z)| < \sigma_0 \tag{2.2}$$

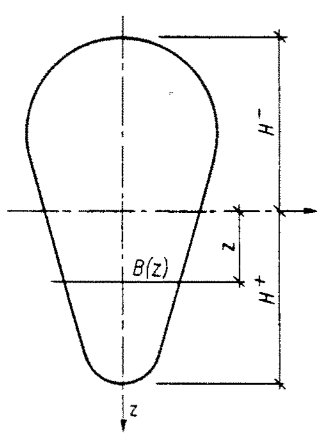

FIG. 1.

assures the elastic behavior of the respective layer. Thus the set of inequalities (2.2) for all z from the domain $(-H^-, H^+)$ defines the elastic locus of the cross-section in the (M, N) plane.

The case when $\rho(z)$ is a piece-wise linear function of z (generally without the requirement of continuity) is of special interest. Let $\rho(z)$ be linear for $z_i < z < z_{i+1}$ for each $i = 0, 1, \ldots, (n-1)$, where $z_0 = -H < z_1 < \ldots < z_n = H^+$. Then to assure that the inequalities (2.2) hold for each z, it is sufficient to satisfy the following set of inequalities:

$$\begin{aligned}
&|\sigma^e(z_0)+\rho(z_0)| < \sigma_0 \\
&|\sigma^e(z_1)+\rho(z_1-0)| < \sigma_0 \\
&|\sigma^e(z_1)+\rho(z_1+0)| < \sigma_0 \\
&|\sigma^e(z_{n-1})+\rho(z_{n-1}-0)| < \sigma_0 \\
&|\sigma^e(z_{n-1})+\rho(z_{n-1}+0)| < \sigma_0 \\
&|\sigma^e(z_n)+\rho(z_n)| < \sigma_0 .
\end{aligned} \tag{2.3}$$

This set contains $2(n+1)$ conditions in the case of continuity up to the number of $4n$ (the case of discontinuity at each z_i). Each of these inequalities (2.3) is linear with respect to M and N, thus the whole set delimits a polygon in the (M, N) plane.

It may be easily seen that such a distribution $\rho(z)$ contains no more than $3n-3$ free parameters.

The condition (2.3) defining the elastic locus can be rewritten in the form:

$$f_i(\alpha_1, \alpha_2, \ldots, \alpha_m)M + g_i(\alpha_1, \alpha_2, \ldots, \alpha_m)N + h_i(\alpha_1, \alpha_2, \ldots, \alpha_m) < 0; \qquad i = 1, 2, \ldots m \tag{2.4}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ denote free parameters mentioned above. It is visible from (2.1) that the functions f_i, g_i, h_i are also linear with respect to the parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$.

If the boundary of an elastic locus contains a point belonging to the yield locus of the section, then the residual stress field associated with that elastic locus is unique. Let the generalized stresses in the limit state be M^* and N^*. The associated total and elastic stress distributions are unique (within the limitations of the uniqueness theorem of limit analysis), thus the residual stress field $\rho(z)$ is unique and it has a discontinuity at some point $z = z^*$. If we now write down the inequalities (2.2) for $z = z^*$, we obtain two opposite inequalities resulting in a single equality. The equality connecting M and N means that the elastic locus considered degenerates into a straight line segment in the (M, N) plane.

Definition: the *initial elastic locus* is the elastic locus under the condition $\rho(z) \equiv 0$. It is easy to see that this elastic locus has the form of a rhomb with the two axes of symmetry coinciding with M and N axes.

After exposing a method of constructing elastic loci through piece-wise linear residual stress distributions in view of further applications, we are going to show some principal theorems regarding the loci.

Theorem I

Any elastic locus, after an appropriate translation may be enclosed by the initial elastic locus.

Proof. Let $P = (M^*, N^*)$ be any point on the initial elastic locus. Also the point $P' = (-M^*, -N^*)$ lies on that locus. It means that there exists a point $z = z^*$ of the cross-section such that the respective stress states are

$$\sigma^P(z^*) = \frac{M^*}{J}z^* + \frac{N^*}{A} = \pm\sigma_0; \qquad \sigma^{P'}(z^*) = -\frac{M^*}{J}z^* - \frac{N^*}{A} = \mp\sigma_0 \tag{2.5}$$

The difference:

$$\sigma^P(z^*) - \sigma^{P'}(z^*) = \pm 2\sigma_0.$$

Now let us consider any elastic locus containing inside it that initial elastic locus translated by a vector (μ, ν), and let Q and Q' denote the points $Q = (M^* + \mu, N^* + \nu)$; $Q' = (-M^* + \mu, -N^* + \nu)$. It is easy to see that for any given residual stress distribution $\rho(z)$ the difference:

$$\sigma^Q(z^*) - \sigma^{Q'}(z^*) = \pm 2\sigma_0;$$

thus both these points, Q and Q' must lie on or outside of any elastic locus. That proves the theorem.

Theorem II

If $\rho_1(z)$, $\rho_2(z)$ are two residual stress fields associated respectively with the elastic loci S_1 and S_2, then the region S_3 of all points P of the form

$$P = \lambda P_1 + (1-\lambda)P_2; \qquad P_1 \in S_1,\ P_2 \in S_2,\quad 0 < \lambda < 1$$

is contained within the elastic locus associated with the residual stress field

$$\rho(z) = \lambda\rho_1(z) + (1-\lambda)\rho_2(z)$$

λ being a constant.

Proof. We have

$$\left|\frac{N_1}{A} + \frac{M_1}{J}z + \rho_1(z)\right| < \sigma_0 \quad \text{for all } (M_1, N_1) = P_1 \in S_1$$

$$\left|\frac{N_2}{A} + \frac{M_2}{J}z + \rho_2(z)\right| < \sigma_0 \quad \text{for all } (M_2, N_2) = P_2 \in S_2$$

at all points of the sections. Thus if

$$N = \lambda N_1 + (1-\lambda)N_2; \quad M = \lambda M_1 + (1-\lambda)M_2,$$

then

$$\left|\frac{N}{A} + \frac{M}{J}z + \rho(z)\right| = \left|\lambda\left[\frac{N_1}{A} + \frac{M_1}{J}z + \rho_1(z)\right]\right.$$
$$\left. + (1-\lambda)\left[\frac{N_2}{A} + \frac{M_2}{J}z + \rho_2(z)\right]\right| \le \lambda\left|\frac{N_1}{A} + \frac{M_1}{J}z + \rho_1(z)\right|$$
$$+ (1-\lambda)\left|\frac{N_2}{A} + \frac{M_2}{J}z + \rho_2(z)\right| \le \sigma_0$$

at all points of the section. This shows that any point $(M, N) = P \in S_3$ lies within the elastic locus S.

3. ELASTIC LOCI FOR RECTANGULAR CROSS-SECTIONS

We shall consider in the examples an arch and a frame with rectangular cross-sections; it is worth while to present some groups of elastic loci for that cross-section. For the sake of simplicity we introduce the following dimensionless quantities:

$$m = \frac{M}{\sigma_0 H^2 B}; \qquad n = \frac{N}{2\sigma_0 H B} \tag{3.1}$$

and we employ the plane (m, n) instead of (M, N). The initial elastic locus is now described by four inequalities:

$$\begin{aligned} n &\leq 1 - \tfrac{3}{2}m; \qquad n \geq -1 + \tfrac{3}{2}m \\ n &\leq 1 + \tfrac{3}{2}m; \qquad n \geq -1 + \tfrac{3}{2}m \end{aligned} \tag{3.2}$$

The simplest families of elastic loci are as follows:

(a) The family of the degenerated elastic loci containing points from the yield locus. Some of them are presented in Fig. 2.

(b) The family associated with the residual stress described in Fig. 3. Here

$$\rho(z) = \begin{cases} -\alpha\sigma_0 - 2\alpha\sigma_0 \dfrac{z}{H} & z \leq 0 \\ -\alpha\sigma_0 + 2\alpha\sigma_0 \dfrac{z}{H} & z \geq 0 \end{cases} \tag{3.3}$$

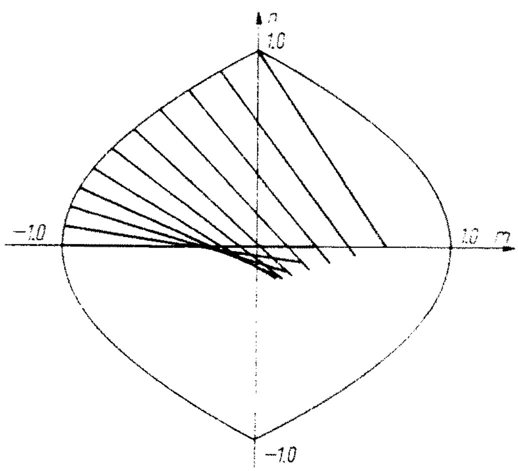

FIG. 2.

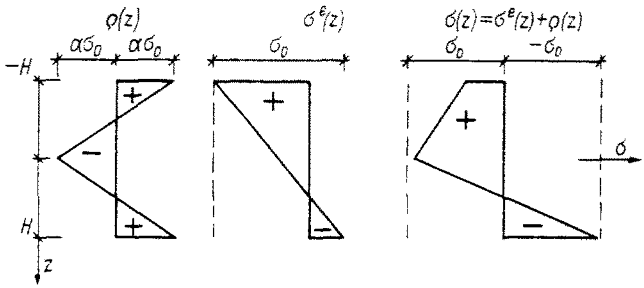

FIG. 3.

Superimposing the elastic stress

$$\sigma^e(z) = \sigma_0\left(n+\frac{3}{2}m\frac{z}{H}\right) \tag{3.4}$$

and using the conditions (2.2) we obtain:

$$\begin{aligned} &n+\alpha \leq 1-\tfrac{3}{2}m; \qquad n+\alpha \geq -1+\tfrac{3}{2}m; \qquad n+\alpha \leq 1 \\ &n+\alpha \leq 1+\tfrac{3}{2}m; \qquad n+\alpha \geq -1-\tfrac{3}{2}m; \qquad n-\alpha \geq -1 \end{aligned} \tag{3.5}$$

(the points which have to be considered being $z = \pm H$ and $z = 0$). Figure 4 shows that only five of those inequalities are important and that the loci for $|\alpha| > \frac{1}{2}$ are not useful.

(c) The family associated with the discontinuous residual stress distribution as shown in Fig. 5. Now

$$\rho(z) = \begin{cases} -2\beta\sigma_0-3\beta\sigma_0\dfrac{z}{H}; & z \leq 0 \\ \\ 2\beta\sigma_0-3\beta\sigma_0\dfrac{z}{H}; & z \geq 0 \end{cases} \tag{3.6}$$

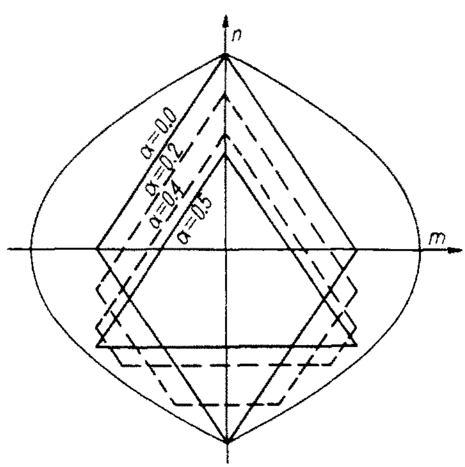

FIG. 4.

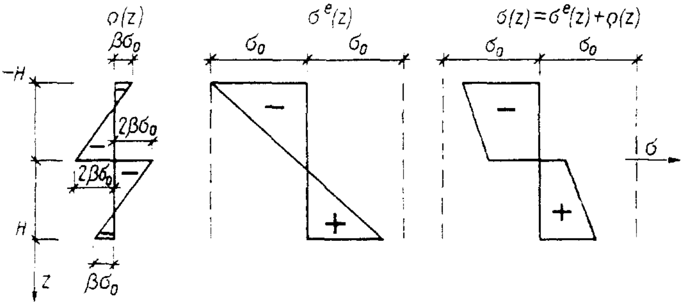

FIG. 5.

The resulting elastic loci are described also by the set of six inequalities (see Fig. 6):

$$
\begin{aligned}
&n \leq 1-\tfrac{3}{2}(m-\beta)\ \text{(a)}; && n \geq -1-\tfrac{3}{2}(m-\beta)\ \text{(d)};\\
&n \leq 1+\tfrac{3}{2}(m-\beta)\ \text{(b)}; && n \leq 1-3|\beta|\ \text{(e)};\\
&n \geq -1+\tfrac{3}{2}(m-\beta)\ \text{(c)}; && n \leq -1+3|\beta|\ \text{(f)}.
\end{aligned}
\tag{3.7}
$$

It is visible that practically $|\beta| \leq \frac{1}{3}$.

Putting $\lambda = 3|\beta|$ this family can be obtained also by applying the theorem II to the initial elastic locus and to the loci from (a) containing the point $m = -1$; $n = 0$ or $m = 1$; $n = 0$.

4. UPPER BOUND

Generally speaking, the upper bound shakedown theorem of Koiter [5] does not need any reformulation in order to be used in terms of the theory employing generalized quantities except possibly for the case where we wish to distinguish between an incremental collapse and a low cycle fatigue.

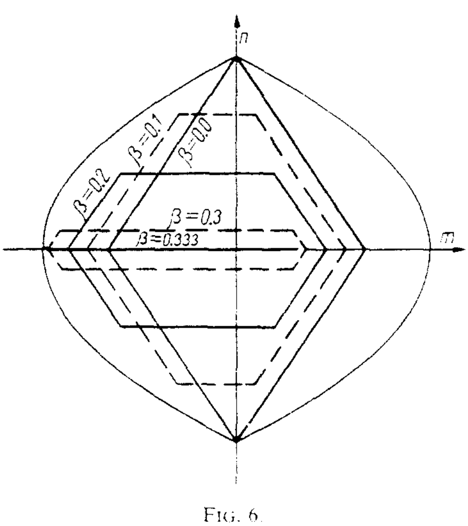

FIG. 6.

The non-adaptation results either in the incremental collapse (when the plastic deformations of the structure grow up without limits), or in the alternating plasticity causing the collapse by plastic fatigue.

The first type of collapse requires a plastic mechanism to arise. This fact was utilized to simplify the computation in papers [14–17].

The alternating plasticity requires the domain of variation of the elastic part of the stress to be sufficiently broad; thus the strictly static analysis suffices to obtain the upper bound (see [14]). The same result may be obtained by using the theorem I together with the appropriately restated Melan theorem. This will be explained in the last two chapters.

Obviously the suitable part of the yield locus in the space of the load parameters may also be used for that purpose.

5. LOWER BOUND

The construction of the lower bound of the shakedown domain will be based on the generalized Melan theorem. This theorem operates with elastic loci and with residual generalized stresses.

If the considered bar structure has k redundants, then the residual generalized stress field (i.e. that which is in equilibrium with vanishing external loads) is described by k parameters $\gamma_1, \gamma_2, \ldots, \gamma_k$ and the residual moments $M^0(\xi)$ as well as the residual axial forces $N^0(\xi)$ may be presented as below, if the parameters $\gamma_1, \gamma_2, \ldots, \gamma_k$ are appropriately chosen.

$$M^0(\xi) = \sum_{i=1}^{k} \gamma_i a_i(\xi); \qquad N^0(\xi) = \sum_{i=1}^{k} \gamma_i b_i(\xi). \tag{5.1}$$

Let us assume that we choose the elastic loci from the l-parametric family which constitutes polygons in (M, N) plane. For this purpose, for example, the elastic loci generated by piece-wise linear residual stress fields, described by the formulae (2.4) can serve. Thus the shakedown requires the following system of inequalities to be satisfied.

$$f_i[\alpha_1(\xi), \alpha_2(\xi), \ldots, \alpha_l(\xi)]M(\xi) + g_i[\alpha_1(\xi), \alpha_2(\xi), \ldots, \alpha_l(\xi)]N(\xi) + h_i[\alpha_1(\xi), \alpha_2(\xi), \ldots, \alpha_l(\xi)] < 0 \tag{5.2}$$

at all cross-sections of the structure.

The total generalized stresses are now as follows:

$$M(\xi) = \sum_{i=1}^{r} p_i M^i(\xi) + M^0(\xi); \qquad N(\xi) = \sum_{i=1}^{r} p_i N^i(\xi) + N^0(\xi) \tag{5.3}$$

[see formulae (1.1)]. When the load program is defined by (1.2) the domain of variation of generalized stresses forms the convex hull of all the following points in the (M, N) plane:

$$\begin{aligned} M(\xi) &= \sum_{i=1}^{r} \beta_i p_i^s M^i(\xi) + M^0(\xi) \\ N(\xi) &= \sum_{i=1}^{r} \beta_i p_i^s N^i(\xi) + N^0(\xi) \end{aligned} \tag{5.4}$$

where $\beta_i = -\delta_i$ or $\beta_i = 1$ and $M^0(\xi)$ and $N^0(\xi)$ are given by (5.1) and do not depend on p_i^s. It may be found that there are only $2r$ important combinations of β–s i.e. $2r$ corners (5.4) which have to be taken into account.

Usually for bar structures (and always if the structure is loaded by concentrated loads) we are able to show the finite set of cross sections $\xi = \xi_1, \xi = \xi_2, \ldots, \xi = \xi_q$ such that if the formulae (5.2) hold for all $\xi_1, \xi_2, \ldots, \xi_q$ then they hold everywhere. Therefore the condition of adaptation of the bar structure under consideration takes the form

$$\begin{aligned} f_i[\alpha_1(\xi_j), \ldots, \alpha_l(\xi_j)]\left[\sum_{m=1}^{r} \beta_m p_m^s M^m(\xi_j) + \sum_{m=1}^{k} \gamma_m a_m(\xi_j)\right] + g_i[\alpha_1(\xi_j), \ldots, \alpha_l(\xi_j)] \\ \times \left[\sum_{m=1}^{r} \beta_m p_m^s N^m(\xi_j) + \sum_{m=1}^{k} \gamma_m b_m(\xi_j)\right] + h_i[\alpha_1(\xi_j), \ldots, \alpha_l(\xi_j)] < 0 \\ i = 1, 2, \ldots, t; \qquad j = 1, 2, \ldots, q. \end{aligned} \tag{5.5}$$

As already mentioned, there are $2r$ important combinations of β–s. Therefore we obtain $t \,.\, q \,.\, 2r$ inequalities (5.5) to satisfy.

Obviously we are interested in finding parameters $\gamma_1, \ldots, \gamma_k, \alpha_l(\xi_j)$ such that the coefficients p_i^s could be as high as possible. To be more precise let $p_2^s, p_3^s, \ldots, p_r^s$ be fixed and now we look for the highest possible value of the parameter p_1^s allowing to satisfy the inequalities (5.5).

In this formulation the problem coincides with one of the fundamental problems of linear programming. Namely with the following one:

Find the highest (lowest) value of the linear form

$$f = c_1 x_1 + c_2 x_2 + \ldots + c_u x_u \tag{5.6}$$

under the conditions:

$$\sum_{j=1}^{u} \varphi_{ij} x_j + \psi_i < 0; \qquad i = 1, 2, \ldots, u \tag{5.7}$$

(see [19]). The solution of that problem is known in the linear algebra and there is no point in recalling it here. The algorithm used may be easily adapted for digital computer calculations.

In our case the form (5.6) reduces to:

$$f = p_1^s$$

and conditions (5.5) play the part of the inequalities (5.7). We employ the parameters $\gamma_1, \ldots, \gamma_k, \alpha_l(\xi_j)$ and the parameter p_1^s as the variables $x_1, x_2, \ldots, x_u$. Thus, we can obtain a single point of the lower bound of the shakedown interaction surface. Such a procedure has to be repeated several times with other values of the parameters $p_2^s, p_3^s, \ldots, p_r^s$ to determine a sufficient number of points of that approximated surface. The convex hull of these points constitutes a lower bound of the shakedown domain.

6. CIRCULAR ARCH

Let us analyse the clamped circular arch (see Fig. 7) of a rectangular cross-section under a two-parameter load, namely under the vertical force P and the bending moment C. For the sake of simplicity we introduce the dimensionless quantities:

$$c = \frac{C}{\sigma_0 H^2 B}; \qquad p = \frac{P}{2\sigma_0 HB}; \qquad \rho = \frac{R}{H} \tag{6.1}$$

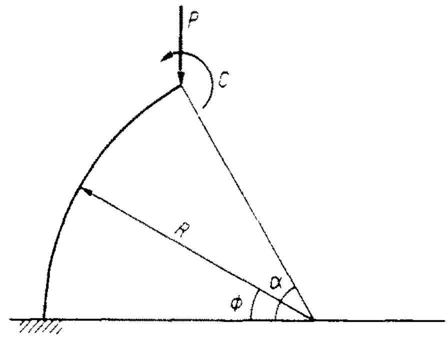

FIG. 7.

The generalized stresses $m(\phi)$, $n(\phi)$ are related to the load parameters c, p by the following formulae

$$\begin{aligned} m(\phi) &= c-2pj(\cos\phi-\cos\alpha) \\ n(\phi) &= -p\cos\phi. \end{aligned} \tag{6.2}$$

We shall consider the case $j = 5$, $\alpha = 60°$ and the load program of the form

$$0 < p < p^s \qquad -c_s < c < c_s. \tag{6.3}$$

The domain of variation of generalized stresses in (m, n) plane constitutes a parallelogram with the following corners:

$$\begin{array}{lllll} \text{corner I} & p = 0; & c = -c_s; & m(\phi) = -c_s; & n(\phi) = 0 \\ \text{corner II} & p = 0; & c = c_s; & m(\phi) = c_s; & n(\phi) = 0 \\ \text{corner III} & p = p_s; & c = -c_s; & m(\phi) = -c_s-10p_s(\cos\phi-\frac{1}{2}); & n(\phi) = -p_s\cos\phi \\ \text{corner IV} & p = p_s; & c = c_s; & m(\phi) = c_s-10p_s(\cos\phi-\frac{1}{2}); & n(\phi) = -p_s\cos\phi. \end{array} \tag{6.4}$$

The formulae (6.2) show that the stress profile is a straight-line segment in the (m, n) plane. Thus in the shakedown analysis it will suffice to take into account only the ends of the arch, namely the cross-sections $\phi = 0$ and $\phi = 60°$. Figure 8 presents the domains of variation of the generalized stresses m, n at those cross-sections, when p and c vary arbitrarily within the limits prescribed by the load program (6.3).

Upper bound

The theorem I implies that the adaptation is impossible if the domain of variation of elastic generalized stresses cannot be contained (at least at one cross-section of the structure) within any translated initial elastic locus.

In our case, if the domains from Fig. 8 have to be enclosed by appropriately translated initial elastic locus (2.7), then all points of those domains have to satisfy the following inequalities [the vector (b, a) denotes the translation of the locus]:

$$\begin{array}{ll} n-a \leq 1-\frac{3}{2}(m-b)\,(\mathrm{a}); & n-a \geq -1+\frac{3}{2}(m-b)\,(\mathrm{c}); \\ n-a \leq 1+\frac{3}{2}(m-b)\,(\mathrm{b}); & n-a \geq -1-\frac{3}{2}(m-b)\,(\mathrm{d}). \end{array} \tag{6.5}$$

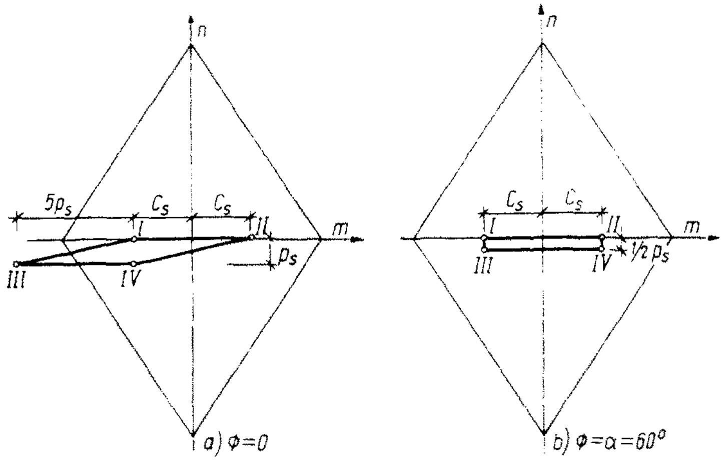

FIG. 8.

By comparing that rhomb with the domains from Fig. 8 we can see that it suffices to satisfy condition (a) at the corner II and condition (d) at the corner III. Thus we obtain:

$$n^{II}-a \leq 1-\tfrac{3}{2}(m^{II}-b); \qquad n^{III}-a \leq -1-\tfrac{3}{2}(m^{III}-b). \tag{6.6}$$

The combination of those inequalities gives:

$$n^{II}-n^{III} \leq 2-\tfrac{3}{2}(m^{II}-m^{III}). \tag{6.7}$$

After substitution of (6.4) for $\phi = 0$ and $\phi = 60°$ we obtain respectively

$$3c_s+8{\cdot}5p_s \leq 2; \qquad 3c_s+0{\cdot}5p_s \leq 2. \tag{6.8}$$

The second condition is unimportant because it may be deduced from the first one.

On the other hand the collapse values (in the sense of limit analysis) can be used also as an upper bound of the shakedown curve in the (c_s, p_s) plane. This gives:

$$c_s \leq 1-5p_s-p_s^2. \tag{6.9}$$

The upper bound, obtained as a combination of the inequalities (6.8) and (6.9) is drawn in Fig. 10 in the dashed line.

Remark. The formula for the case of the sandwich cross-section analogous to (6.7) was derived by Hodge and Kalinowski in [14] directly from the definition of cyclic collapse.

Lower bound

We shall operate first with the family of elastic loci from Fig. 7 as described by the formulae (3.7). It is not difficult to conclude that when the elastic locus appropriate for the cross-section $\phi = 0$ is found, then also the elastic loci for $0 < \phi < \alpha = 60°$ exist in the considered family.

Let the value of the parameter β appropriate for $\phi = 0$ be β^*. Then the following inequalities must hold:

$$n^{III} \geq -1-\tfrac{3}{2}(m^{III}+\beta^*); \qquad n^{II} \geq -1+\tfrac{3}{2}(m^{II}+\beta^*); \qquad n^{III} \geq -1+3\beta^* \text{ where } 0 \leq \beta^* \leq \tfrac{1}{2}.$$

These give:

$$\tfrac{3}{2}c_s+8{\cdot}5p_s \leq 1+\tfrac{3}{2}\beta^*; \qquad c_s+\beta^* \leq \tfrac{2}{3}; \qquad p_s \leq 1-3\beta^*. \tag{6.10}$$

It is easy to find that the point $c_s = \frac{2}{3}$, $p_s = 0$ belongs to the shakedown domain because its coordinates satisfy (6.10) if $\beta^* = 0$. Also the values $c_s = 0{\cdot}34$, $p_s = 0{\cdot}11$, $\beta^* = 0{\cdot}2967$ satisfy the inequalities (6.10) as well as the values:

$$c_s = 0, \qquad p_s = \tfrac{1}{6}, \qquad \beta^* = \tfrac{5}{18}.$$

By connecting the three points by straight lines we obtain the lower bound of the shakedown curve we are looking for.

The lower bound in the neighbourhood of $c_s = 0$ may be improved. Namely, we construct a new elastic locus using the theorem II. We take as the two elastic loci the elastic locus from the Fig. 4, described by (2.10) for $\alpha = \frac{1}{2}$ and the degenerated elastic locus from Fig. 2 with one end $m^* = -0{\cdot}99$, $n^* = -0{\cdot}10$. The resulting domain for the coefficient $\lambda = \frac{1}{2}$ was shown in Fig. 9. After computation we find that the point $c_s = 0$, $p_s = 0{\cdot}184$

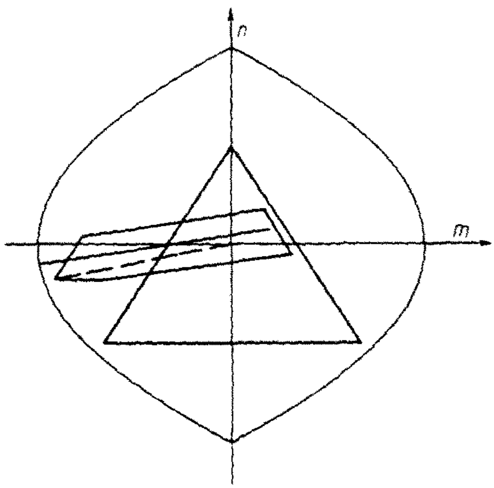

FIG. 9.

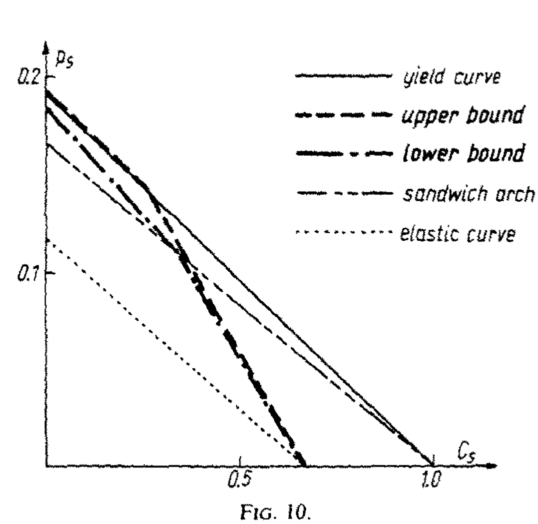

FIG. 10.

belongs also to the shakedown domain. Therefore the lower bound based on the three points:

$$c_s = \tfrac{2}{3} \qquad c_s = 0{\cdot}34 \qquad c_s = 0{\cdot}0$$

$$p_s = 0{\cdot}0 \qquad p_s = 0{\cdot}11 \qquad p_s = 0{\cdot}184$$

has the form of the dot-and-dash line in Fig. 10.

The difference between the upper and the lower bound is rather small. For the sake of comparison the elastic interaction curve and the yield curve have been drawn also in Fig. 10. The exact solution of the shakedown problem for the sandwich arch (i.e. of the same yield moment and the same yield axial load) is also presented.

7. PORTAL FRAME UNDER HEAVY AXIAL LOADS

We consider the portal frame as in Fig. 11, of rectangular cross-section, subjected to the load program:

$$0 < p < p_s; \qquad -h_s < h < h_s \tag{7.1}$$

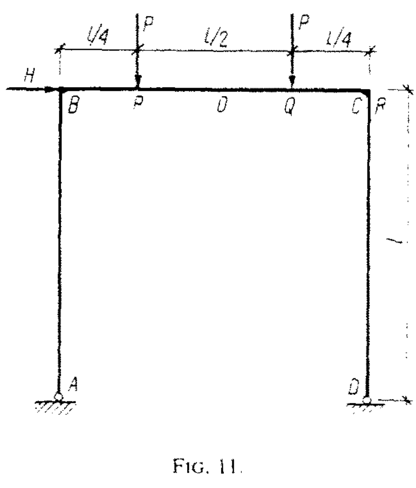

FIG. 11.

$$p = \frac{P}{2\sigma_0 HB}; \qquad h = \frac{H}{2\sigma_0 HB}; \qquad s = \frac{S}{2\sigma_0 HB}; \qquad \lambda = \frac{l}{H} = 0 \tag{7.2}$$

S denotes the residual horizontal reaction at points A and D, directed inward.

It may be found that the most dangerous sections are at points P (in the beam), C (in the beam), R (in the column). The generalized stresses at those cross-sections are

$$\begin{aligned} \text{at } P: \quad & m_P = 2{\cdot}75\,p + 5\,h - 20\,s \\ & n_P = -0{\cdot}1125\,p - 0{\cdot}5\,h - s \\ \text{at } C: \quad & m_C = -2{\cdot}25\,p - 10\,h - 20\,s \\ & n_C = -0{\cdot}1125\,p - 0{\cdot}5\,h - s \\ \text{at } R: \quad & m_R = -2{\cdot}25\,p - 10\,h - 20\,s \\ & n_R = -p - h. \end{aligned} \tag{7.3}$$

The domain of variation of generalized stresses is a parallelogram in (m, n) plane, with the corners:

$$\begin{array}{lll} \text{corner I} & p = 0 & h = -h_s \\ \text{corner II} & p = 0 & h = h_s \\ \text{corner III} & p = p_s & h = -h_s \\ \text{corner IV} & p = p_s & h = h_s \end{array} \tag{7.4}$$

It may be found that for all cross-sections P, C, R only the corners I and IV are important if we employ the family of elastic loci from the Fig. 6.

Upper bound

Using the same reasoning as in the case of the arch, we obtain in the following form the conditions which assure that no alternating plasticity can occur.

$$n_P^{\mathrm{I}} - n_P^{\mathrm{IV}} - \tfrac{3}{2}(m_P^{\mathrm{I}} - m_P^{\mathrm{IV}}) \le 2; \quad n_C^{\mathrm{I}} - n_C^{\mathrm{IV}} + \tfrac{3}{2}(m_C^{\mathrm{I}} - m_C^{\mathrm{IV}}) \le 2; \quad n_R^{\mathrm{I}} - n_R^{\mathrm{IV}} + \tfrac{3}{2}(m_R^{\mathrm{I}} - m_R^{\mathrm{IV}}) \le 2. \tag{7.5}$$

These inequalities give, according to (7.3)

$$h_s \le 0{\cdot}125 - 0{\cdot}2648 p_s; \quad h_s \le 0{\cdot}0645 - 0{\cdot}1125 p_s; \quad h_s \le 0{\cdot}0625 - 0{\cdot}13675 p_s. \tag{7.6}$$

The last of those conditions is valid. We complete the upper bound by the collapse value for $h_s = 0$ which is

$$p_0 = 0{\cdot}3725 \tag{7.7}$$

i.e. $p_s = 0{\cdot}3725$. Figure 12 presents these bounds in dashed line.

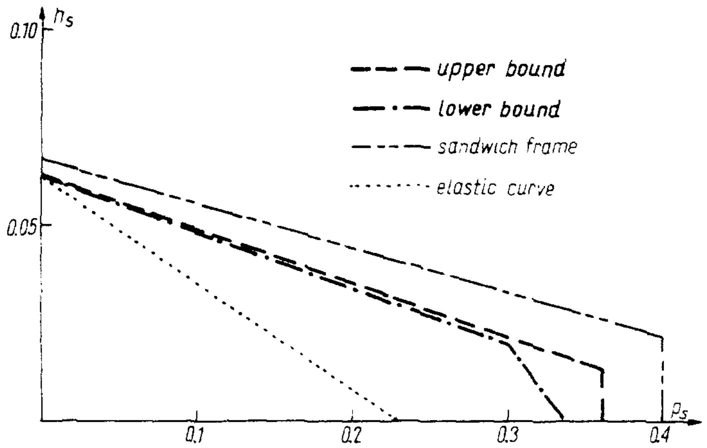

Fig. 12.

Lower bound

We employ the family of elastic loci from Fig. 6. It may be found that to contain the generalized stresses (7.3) within an appropriate elastic locus from that family it suffices to satisfy

$$\begin{array}{lll} \text{at } P: & \text{corner I} & \text{satisfies (b) and (e) from (3.7)} \\ & \text{corner IV} & \text{satisfies (c) and (f) from (3.7)} \\ \text{at } C: & \text{corner I} & \text{satisfies (a) and (e) from (3.7)} \\ & \text{corner IV} & \text{satisfies (d) and (f) from (3.7)} \\ \text{at } R: & \text{corner I} & \text{satisfies (a) and (e) from (3.7)} \\ & \text{corner IV} & \text{satisfies (d) and (f) from (3.7).} \end{array} \tag{7.8}$$

Thus, we obtain eighteen inequalities. After substitution of (7.3) into (7.8) we eventually obtain

$$\begin{aligned} &8h_s \leq 1+1{\cdot}5\beta_P-29s \\ &0{\cdot}5h_s < 1-3\beta_P+s \\ &4{\cdot}1375p_s+8h_s < 1-1{\cdot}5\beta_P+29s \\ &0{\cdot}1125p_s+0{\cdot}5h_s < 1-3|\beta_P|-s \\ &\quad 15{\cdot}5h_s < 1-1{\cdot}5\beta_C+31s \\ &\quad 0{\cdot}5h_s < 1-3|\beta_C|+s \\ &\quad 3{\cdot}4875p_s+15{\cdot}5h_s < 1+1{\cdot}5\beta_C-31s \\ &\quad 0{\cdot}1125p_s+0{\cdot}5h_s < 1-3|\beta_C|-s \\ &16h_s < 1-1{\cdot}5\beta_R+30s \\ &h_s < 1-3|\beta_R| \\ &4{\cdot}375p_s+16h_s < 1+1{\cdot}5\beta_R-30s \\ &p_s+h_s < 1-3|\beta_R|. \end{aligned} \tag{7.9}$$

It may be checked that putting

$$p_s = 0{\cdot}3; \qquad h = 0{\cdot}02 \tag{7.10}$$

we are able to satisfy all the inequalities (7.9) if

$$s = -0{\cdot}01, \qquad 0{\cdot}221 < \beta_R < 0{\cdot}253, \qquad -0{\cdot}320 < \beta_P < 0{\cdot}284, \qquad 0{\cdot}038 < \beta_C < 0{\cdot}253 \tag{7.11}$$

Also the following values

$$\begin{gathered} p_s = 0{\cdot}336; \qquad h_s = 0{\cdot}00 \\ \beta_P = -0{\cdot}30; \qquad \beta_C = 0; \qquad \beta_R = 0{\cdot}1808; \qquad s = -0{\cdot}0253 \end{gathered} \tag{7.12}$$

satisfy all the inequalities (7.9). The point $p_s = 0$; $h_s = 0{\cdot}0625$ obviously belongs to the shakedown domain because it belongs to the domain of the perfectly elastic response of the structure (see Fig. 12). This domain of elastic response may also be obtained from (7.9) by putting

$$s = \beta_P = \beta_C = \beta_R = 0.$$

The resulting lower bound has been shown in Fig. 12 in a dot-and-dash line.

For the sake of comparison the exact shakedown curve for that frame with no influence of the axial force N on the yielding of the cross-section is presented in Fig. 12. This shows that in the case of the structure considered the influence of the axial force is rather important.

8. CONCLUSIONS

The aim of the paper was to present a general method allowing to construct a lower bound of shakedown in the case of frames and arches of an arbitrary cross-section for which the influence of axial forces cannot be neglected. The construction of an upper bound was equally studied.

More complex structures have to be calculated with the use of computers. It has been shown that the problem considered may be reduced effectively to a problem of linear programming.

Both examples show that generally:

1. Neither the collapse load nor the perfectly elastic calculation give the proper value of shakedown load parameters.
2. The use of the idealized sandwich cross-section (which is frequent and useful in limit analysis) does not give a safe estimation of lower or upper bounds of shakedown (as in Section 6).
3. Not only in arches but also in frames with higher axial loads the influence of the axial force N may be important and in the frame from Section 7, for instance, the difference can approximate 33 per cent.

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Абстракт—Конструкции, вообще, подвергают программам нагружения более сложным чем рассматриваемые в теории предельного равновесия. Теория приспособления представляет общие теоремы позволяющие оценить приспособляемость конструкций к данной программе наагружения. На основании предыдущих результатов (22) в настоящей работе представлен приближенный метод анализа стержневых конструкций любого поперечного сечения в которых влиянием осевых сил пренебречь нельзя. Поставленная проблема сводится к задаче линейного программирования. Представлены численные результаты для портальной рамы и для круговой арки прямоугольного поперечного сечения.